tate facile perſpecturus foret; jam vero, quoniam egregium illud Rei Medicæ Lumen amiſimus, eadem aliis Eruditis perpendenda ſimul proponimus & dijudicanda. Tibi præſertim, Vir Doctiſſime, cujus auctoritatem & ille plurimi fecit, & nos præcipuam habemus, Judici ſimul integerrimo & maxime idoneo, totam iſtam diſputationem lubentiſſime ſubjicimus.

III. *Methodus Differentialis* Newtoniana *Illuſtrata.* *Authore* Jacobo Stirling, *è Coll. Balliol.* Oxon.

ARithmeticæ pars præcipua conſiſtit in inveniendâ in numeris quantitate quâcunque determinatâ; cum vero quantitatum & numerorum natura non patiatur ut omnes quantitates exhibeantur in numeris accurate, neceſſe habemus ad Approximationes confugere. Hoc eſt, ubi quantitatum valores mathematice accurati neqeunt obtineri, quærendi ſunt ii qui ab accuratis diſtant minus datâ quâvis differentiâ.

Quicquid hâc de re à Veteribus ad nos pervenit, vel eſt particulare, ut Methodus eorum reducendi Æquationes Quadraticas; vel ſaltem uſibus generalibus male deſtinatum, ut Methodus Exhauſtionum. *Vieta* quidem primus erat qui aliquid generale in hâc arte aſſequutus eſt: quippe invenit methodum reducendi Æquationes Rationales, quæ ſolæ tunc in uſu erant. In hâc acquievêre omnes Geometræ ex ejus temporibus uſque ad ea *Newtoni.* Hic ex Interpolationibus primo pervenit ad Series: quas poſtea ad reductionem Æquationum omnium omnino generum univerſaliter applicuit. Hæc autem methodus procedit per quantitatum naſcentium & evaneſcentium rationes primas & ultimas, ſeu ſi ita loqui liceat, per quantitatum coincidentium

cidentium differentias infinite parvas. Sed & ulterius promovit *Newtonus* hanc methodum; docuitque qua ratione approximandum ſit ad quantitates quæ determinantur per regularem ſeriem terminorum, non per Æquationem ut vulgo fit. Atque ſic poſuit fundamenta calculi hujus Differentialis, qui procedit per quantitatum differentias cujuſcunque magnitudinis: ideoque eſt methodo Serierum univerſalior. Per haſce artes *Newtonianas* univerſa doctrina Approximationum reducitur ad ſolutionem Problematis, *Invenire Lineam Geometricam quæ per data quotcunque puncta tranſibit.* Ex hujus inquam ſolutione inveniuntur radices Æquationum quarumcunque, & etiam quantitates quarum relationes ad alias datas per nullas Æquationes hactenus notas poſſunt exprimi. Exiſtimo igitur *Newtonum* perduxiſſe methodum Approximandi ad ſummum perfectionis faſtigium; dum ex unico ſimpliciſſimo principio totam hanc doctrinam longe lateque patentem deducit. Quapropter credendum eſt animum *Newtoni* non ſatis perſpectum fuiſſe iis, qui ejus methodos appellant particulares, & alias tanquam ſuas & ſolas genuinas atque generales venditant, quæ aliæ non erant quam Corollaria facillima à *Newtonianis.*

Author noſter, in Epiſtola ad *Oldenburgum, Octob.* 24. 1676. data, mentionem fecit de methodo expeditâ ducendi Lineam Parabolicam per data quotcunque puncta; qua dixit ſe uſum fuiſſe ubi Series ſimplices non ſunt ſatis tractabiles. Et hanc methodum primo publicavit in Lemmate quinto Libri tertii *Principiorum.* Atque in Lectionibus publicis, circa idem tempus quo dicta Epiſtola ſcripta eſt, *Cantabrigiæ* habitis, expoſuit modum generalem determinandi Curvas cujuſcunque generis quæ tranſibunt per totidem data puncta quot earum natura patitur. Hæ Lectiones ſub titulo *Arithmeticæ Univerſalis* anno 1707. publicatæ ſunt, ubi habetur

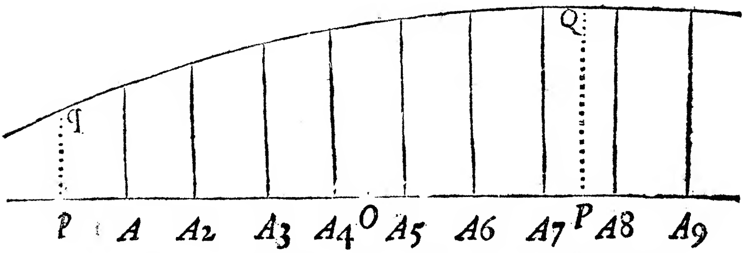
betur methodus exemplis illustrata in sectionibus Conicis. Anno vero 1711. tandem prodiit, inter alios ejusdem Authoris tractatus, ipsa Methodus Differentialis plenius quam ante exposita, cum fundamento ejus demonstrato

Archimedes in methodo Exhaustionum, *Cavallerius* in methodo Indivisibilium, & *Wallisius* noster in Arithmetica Infinitorum, posuerunt fundamenta doctrinæ de determinanda quantitate quæsitâ per locum quem obtinet inter terminos in data Serie: at qua ratione approximandum esset ad valores quantitatum sic determinatarum. horum nemo docuit; Hoc primus & solus perfecit *Newtonus:* atque exinde haud parum ampliata est universa Analysis. Nam sicut ante hoc inventum, ea Problemata Arithmetica sola pro solutis habebantur, ubi relatio quantitatis quæsitæ ad alias datas definiebatur Æquatione, jam pro solutis habenda sunt non minus ea, in quibus quantitas quæsita locum datum sortitur inter terminos datæ Seriei; siquidem numeri desiderati non minus accurate obtinentur per Methodum Differentialem, quam per extractionem Radicum: hisce vero habitis, parum interest quomodo ad eos deventum est. Et experientia multiplex docuit, quod plurima Problemata ad Æquationes ægre deducuntur, dum ad methodum Differentialem facillime. Qualis est ex multis aliis toties decantata Circuli Quadratura; quam tam perfectam, mea opinione, *Wallisius* in Arithmetica Infinitorum exhibuit quam *Archimedes* illam Parabolæ.

Propositio

Propositio.

Invenire Lineam Parabolicam quæ transibit per extrema Ordinatorum quotcunque æquidistantium.



B B2 B3 B4 B5 B6 B7 B8

C C2 C3 C4 C5 C6 C7

D D2 D3 D4 D5 D6

E E2 E3 E4 E5

F F2 F3 F4

G G2 G3

H H2

I

Casus Primus.

Designent *A, A2, A3, A4, A5, A6, A7, A8, A9, &c.* Ordinatas æquidistantes insistentes Abscissæ in dato angulo. Collige earum differentias *B, B2, B3, B4, B5, B6, B7, B8, &c.* harumque differentias *C, C2, C3, C4, C5, C6, C7, &c.* harumque differentias *D, D2, D3, D4, D5, D6, &c.* harumque differentias *E, E2, E3, E4, E5, &c.* harumque *F, F2, F3, F4, &c.* Et sic porro. Differentiæ autem colligi debent auferendo

ferendo priores semper de posterioribus. Hoc est ponendo $B = A2 - A$, $B2 = A3 - A2$, $B3 = A4 - A3$, $B4 = A5 - A4$, $B5 = A6 - A5$, &c. Tum $C = B2 - B$, $C2 = B3 - B2$, $C3 = B4 - B3$, $C4 = B5 - B4$, &c. deinde $D = C2 - C$, $D2 = C3 - C2$, $D3 = C4 - C3$, &c. Et similiter sunt omnes differentiæ sequentes colligendæ. Vel sint $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$, &c. æquales A, $A2$, $A3$, $A4$, $A5$, $A6$, $A7$, &c. Eritque $A = \alpha$, $B = \beta - \alpha$, $C = \gamma - 2\beta + \alpha$, $D = \delta - 3\gamma + 3\beta - \alpha$, $E = \varepsilon - 4\delta + 6\gamma - 4\beta + \alpha$, $F = \zeta - 5\varepsilon + 10\delta - 10\gamma + 5\beta - \alpha$, $G = \eta - 6\zeta + 15\varepsilon - 20\delta + 15\gamma - 6\beta + \alpha$, &c. In hisce valoribus numerales Coefficientes ipsorum α, $\beta, \gamma, \delta, \varepsilon$ &c. generantur ut in dignitatibus integris Binomii $\overline{1-z}|^0$, $\overline{1-z}|^1$, $\overline{1-z}|^2$, $\overline{1-z}|^3$, $\overline{1-z}|^4$, &c. Scribendo numeros 1, 2, 3, 4, 5, &c. in Serie $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$ &c. successive pro n. Sit jam PQ quælibet Ordinata reliquis intermedia, & AP ejus distantia ab Ordinata prima A appelletur z, tum erit

$$
\begin{aligned}
PQ = A + {}& \\
& B \times \frac{z}{1} + \\
& C \times \frac{z}{1} \times \frac{z-1}{2} + \\
& D \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} + \\
& E \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} + \\
& F \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} + \\
& G \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} \times \frac{z-5}{6} + \text{\&c.}
\end{aligned}
$$

Adeoque

Adeoque signum ipsius z mutandum est, quando PQ cadit ad alteras partes Ordinatæ primæ, ut pq.

Casus Secundus.

Sit jam $A5$ Ordinata in medio omnium; pone $A = B4 + B5$, $B = D3 + D4$, $C = F2 + F3$, $D = H + H2$, &c. & $a = C4$, $b = E3$, $c = G2$, $d = I$, &c. id est, si sint $A6 = \alpha$, $A7 = \beta$, $A8 = \gamma$, $A9 = \delta$, &c. $A4 = \kappa$, $A3 = \lambda$, $A2 = \mu$, $A = \nu$, &c. Pone $A = \alpha - \kappa$, $B = \beta - 2\alpha + 2\kappa - \lambda$, $C = \gamma - 4\beta + 5\alpha - 5\kappa + 4\lambda - \mu$, $D = \delta - 6\gamma + 14\beta - 14\alpha + 14\kappa - 14\lambda + 6\mu - \nu$, &c. $a = \alpha - 2A5 + \kappa$, $b = \beta - 4\alpha + 6A5 - 4\kappa + \lambda$, $c = \gamma - 6\beta + 15\alpha - 20A5 + 15\kappa - 6\lambda + \mu$, $d = \delta - 8\gamma + 28\beta - 56\alpha + 70A5 - 56\kappa + 28\lambda - 8\mu + \nu$, &c. Et dicatur $A5P$, z, tum erit

$$PQ = A5 + \frac{Az + azz}{1.2} +$$
$$\frac{2Bz + bzz}{1.2} \times \frac{zz-1}{3.4} +$$
$$\frac{3Cz + czz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} +$$
$$\frac{4Dz + dzz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} \times \frac{zz-9}{7.8} +$$
$$\frac{5Ez + ezz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} \times \frac{zz-9}{7.8} \times \frac{zz-16}{9.10} +$$
&c.

Casus Tertius.

Sint jam $A4$, $A5$, Ordinatæ duæ in medio omnium. Pone $A = \frac{A4 + A5}{2}$, $B = \frac{C3 + C4}{2}$, $C = \frac{E2 + E3}{2}$, $D =$

 $G + G2$

$\frac{G+G2}{2}$, &c. $a=B4$, $b=D3$, $c=F2$, $d=H$, &c. Vel sint $A5=\alpha$, $A6=\beta$, $A7=\gamma$, $A8=\delta$, &c. $A4=\varkappa$, $A3=\lambda$, $A2=\mu$, $A=\nu$, &c. Deinde erunt $2A=\alpha+\varkappa$, $2B=\beta-\alpha-\varkappa+\lambda$, $2C=\gamma-3\beta+2\alpha+2\varkappa-3\lambda+\mu$, $2D=\delta-5\gamma+9\beta-5\alpha-5\varkappa+9\lambda-5\mu+\nu$, &c. Et $a=\alpha-\varkappa$, $b=\beta-3\alpha+3\varkappa-\lambda$, $c=\gamma-5\beta+10\alpha-10\varkappa+5\lambda-\mu$, $d=\delta-7\gamma+21\beta-35\alpha+35\varkappa-21\lambda+7\mu-\nu$, &c. Et sit O punctum medium inter $A4$, $A5$, atque appelletur OP, z; eritque Ordinata

$$PQ=\frac{A+az}{4^0}+$$
$$\frac{3B+bz}{4^1}\times\frac{4zz-1}{2\cdot 3}+$$
$$\frac{5C+cz}{4^2}\times\frac{4zz-1}{2\cdot 3}\times\frac{4zz-9}{4\cdot 5}+$$
$$\frac{7D+dz}{4^3}\times\frac{4zz-1}{2\cdot 3}\times\frac{4zz-9}{4\cdot 5}\times\frac{4zz-25}{6\cdot 7}+$$
$$\frac{9E+ez}{4^4}\times\frac{4zz-1}{2\cdot 3}\times\frac{4zz-9}{4\cdot 5}\times\frac{4zz-25}{6\cdot 7}\times\frac{4zz-49}{8\cdot 9}+\text{\&c.}$$

In hisce duobus etiam casibus z est negativa, quando Ordinata PQ cadit ad alteras partes initii Abscissæ. Et in omnibus tribus casibus distantia communis Ordinatarum ponitur unitas.

Omnes tres casus demonstrantur facillime per calculum. In casu primo pro PQ scribo successive α, β, γ, δ, ε, &c. & pro z interea 0, 1, 2, 3, 4, &c. quæ sunt longitudines Abscissæ ordine sequentes; & provenient æquationes

$$\alpha=A,\ \beta=A+B,\ \gamma=A+2B+C,\ \delta=A+3B+3C+D,$$
$$\varepsilon=A+4B+6C+4D+E,\ \text{\&c.}$$

$\beta-\alpha$

$\beta - \alpha = B$, $\gamma - \beta = B + C$, $\delta - \gamma = B + 2C + D$, $\varepsilon - \delta = B + 3C + 3D + E$, &c.

$\gamma - 2\beta + \alpha = C$, $\delta - 2\gamma + \beta = C + D$, $\varepsilon - 2\delta + \gamma = C + 2D + E$, &c.

$\delta - 3\gamma + 3\beta - \alpha = D$, $\varepsilon - 3\delta + 3\gamma - \beta = D + E$, &c.

$\varepsilon - 4\delta + 6\gamma - 4\beta + \alpha = E$, &c.

Hæ Æquationes, capiendo earum differentias, nullo labore resolvuntur, uti videre est. Et dant eosdem ipsorum A, B, C, D, &c. valores, qui antea positi sunt in solutione. Et ad eundem modum demonstrantur casus duo reliqui.

Harum trium serierum unaquæque converget ad valorem Ordinatæ PQ, ubi Ordinatarum datarum differentiæ sunt justæ magnitudinis. At ubi non convergunt, aliæ artes adhibendæ sunt. Sed impræsentiarum de hujus Propositionis usu pauca adjiciamus.

Designent α, β, γ, δ, ε, ζ, η, θ, $\varkappa$, λ, &c. terminos quoscunque æquidistantes, quorum differentiæ sunt perexiguæ; & relationes quas inter se obtinent definientur quamproxime per Æquationes sequentes, quæ oriuntur capiendo differentias & differentias differentiarum continuò, & ponendo eas æquales nihilo.

$$\alpha - \beta = 0$$
$$\alpha - 2\beta + \gamma = 0$$
$$\alpha - 3\beta + 3\gamma - \delta = 0$$
$$\alpha - 4\beta + 6\gamma - 4\delta + \varepsilon = 0$$
$$\alpha - 5\beta + 10\gamma - 10\delta + 5\varepsilon - \zeta = 0$$
$$\alpha - 6\beta + 15\gamma - 20\delta + 15\varepsilon - 6\zeta + \eta = 0$$
$$\alpha - 7\beta + 21\gamma - 35\delta + 35\varepsilon - 21\zeta + 7\eta - \theta = 0$$
$$\alpha - 8\beta + 28\gamma - 56\delta + 70\varepsilon - 56\zeta + 28\eta - 8\theta + \varkappa = 0$$
$$\alpha - 9\beta + 36\gamma - 84\delta + 126\varepsilon - 126\zeta + 84\eta - 36\theta + 9\varkappa - \lambda = 0.$$

&c.

Hæc

Hæc Tabula in usum reservanda est, ut consulatur quoties opus sit. Quod autem hæ Æquationes vel obtinent accurate, vel ad verum approximant, ubi differentiæ terminorum sunt parvæ, patet ex demonstratione casus primi Propositionis.

Assumatur quælibet Series $\frac{1}{101}$, $\frac{1}{102}$, $\frac{1}{103}$, $\frac{1}{104}$, $\frac{1}{105}$, $\frac{1}{106}$, &c. Et quæratur terminus qui stat proximus ante $\frac{1}{101}$: patet quod ille est $\frac{1}{100}$; videamus ergo qualem hæc methodus exhibebit eundem. Repræsentet α terminum quæsitum, eritque

$\frac{1}{101}=\beta=0099,0099,0099,0$,	Æquatio 1ma dat α	0099,0099,0099,0,
$\frac{1}{102}=\gamma=0098,0392,1568,7$,	2da	0099,9805,8629,3,
$\frac{1}{103}=\delta=0097,0873,7864,1$,	3tia	0099,9994,3455,0,
$\frac{1}{104}=\varepsilon=0096,1538,4615,4$,	4ta	0099,9999,7824,8,
$\frac{1}{105}=\zeta=0095,2380,9523,8$,	5ta	0099,9999,9895,8,
$\frac{1}{106}=\eta=0094,3396,2264,2$.	6ta	0099,9999,9993,1.

Patet ergo quod hæc methodus continue approximat. Si terminorum differentiæ fuissent minores, valores accessissent citius ad verum, & contra tardius quando differentiæ sunt majores. Hinc si in Tabulis numericis desit terminus, potest is per hanc methodum inseri.

Hoc modo etiam prodeunt ipsissimæ Series Speciosæ, quæ per alias methodos prodire solent. Proponatur $\overline{1+zz}|^{-1}$ Ordinata Curvæ quadrandæ: Ea est prima in serie regulari $\overline{1+zz}|^{-1}$, $\overline{1+zz}|^{0}$, $\overline{1+zz}|^{1}$, $\overline{1+zz}|^{2}$, $\overline{1+zz}|^{3}$, &c. Ordinatarum, quæ omnes præter primam dant suas areas z, $z+\frac{1}{3}z^3$, $z+\frac{2}{3}z^3+\frac{1}{5}z^5$, $z+\frac{3}{3}z^3+\frac{3}{5}z^5+\frac{1}{7}z^7$, &c. constituentes novam seriem cujus primus terminus erit Area quæsita: quæ ideo invenietur ponendo pro eâ α, & pro reliquis in suo Ordine β, γ, δ, ε, &c. Prima Æquatio dat $\alpha=z$, secunda $\alpha=z-\frac{1}{3}z^3$, tertia $\alpha=z-\frac{1}{3}z^3+\frac{1}{5}z^5$, quarta $\alpha=z-\frac{1}{3}z^3+\frac{1}{5}z^5-\frac{1}{7}z^7$, &c.

&c. Eſt ergo univerſim area quæſita $z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \frac{1}{9}z^9 - \frac{1}{11}z^{11}$ &c. Eſtque hæc Series arcus ad Tangentem z, in circulo radium habente unitati æqualem. Eam invenit *Jacobus Gregorius* noſter, & cum *Collinio* communicavit initio anni 1671. à quo, mediante *Oldenburgo* ad *Leibnitium* delata eſt.

Sit jam &c, e, d, c, b, a, P, α, β, γ, δ, ε, &c. Series utrinque excurrens in infinitum, ubi dantur omnes termini præter P in medio omnium. Sit $A = \alpha + a$, $B = \beta + b$, $C = \gamma + c$, $D = \delta + d$, $E = \varepsilon + e$, &c. atque erit

$$P = \frac{A}{2} + \frac{A-B}{6} + \frac{5A-8B+3C}{60} + \frac{7A-14B+9C-2D}{140} + \frac{42A-96B+81C-32D+5E}{1260} + \frac{66A-165B+165C-88D+25E-3F}{2772} + \frac{429A-1144B+1287C-832D+325E-72F+7G}{24024} + \&c.$$

Inveſtigatur hæc Series ex Æquationibus, excerpendo alternas in quibus numerus terminorum eſt impar. Nam earum differentiæ relinquent terminos in hac Serie; quæ itaque ad libitum produci poteſt.

Sit $\overline{1+z}|^{-1}$ Ordinata Hyperbolæ, & quæratur Area ejus quæ jacet ſupra Abſciſſam z, quando ea evadit unitas. Hæc Ordinata eſt media in Serie Ordinatarum

rum, &c. $\overline{1+z}|^{-5}$, $\overline{1+z}|^{-4}$, $\overline{1+z}|^{-3}$, $\overline{1+z}|^{-2}$, $\overline{1+z}|^{-1}$, $\overline{1+z}|^{0}$, $\overline{1+z}|^{1}$, $\overline{1+z}|^{2}$, $\overline{1+z}|^{3}$ &c. æquidiſtantium, hinc inde excurrente in infinitum. Adeoque Areæ ab hiſce Ordinatis genitæ conſtituent ſeriem conſimilem, cujus medius terminus erit Area quæſita; quæ proinde obtinebitur per Seriem modo expoſitam. Quando z eſt unitas, ut in caſu præſente, areæ curvarum evadunt &c. $\frac{15}{64}$, $\frac{7}{24}$, $\frac{3}{8}$, $\frac{1}{2}$, & 1, $\frac{3}{2}$, $\frac{7}{3}$, $\frac{15}{4}$, &c. Hinc eſt $A = 1 + \frac{1}{2} = \frac{3}{2}$, $B = \frac{3}{2} + \frac{3}{8} = \frac{15}{8}$, $C = \frac{7}{3} + \frac{7}{24} = \frac{21}{8}$, $D = \frac{15}{4} + \frac{15}{64} = \frac{255}{64}$, &c. Hiſce in Serie ſubſtitutis, prodit P, id eſt, area Hyperbolæ, $\frac{3}{4} - \frac{3}{48} + \frac{3}{480} - \frac{3}{4480} +$ &c. id eſt, $\frac{3}{4} - \frac{A}{4 \cdot 3} - \frac{2B}{4 \cdot 5} - \frac{3C}{4 \cdot 7} - \frac{4D}{4 \cdot 9} - \frac{5E}{4 \cdot 11} -$ &c. Ubi jam A, B, C D, &c. more *Newtoniano*, deſignant terminos in ſuo ordine ab initio. Calculum appono.

TERMINI.

Affirmativi	*Negativi.*
7500,0000,0000,0000,0	0625,0000,0000,0000,0
62,5000,0000,0000,0	6,6964,2857,1428,5
7440,4761,9047,6	845,5086,5800,8
97,5586,9130,8	11,3818,4731,9
1,3390,4086,1	1585,7062,8
188,7745,5	22,5708,7
2,7085,0	3260,2
393,4	47,5
5,7	7
+7563,2539,3930,7494,1	—0631,7821,3370,8041,1

Summam negativam ſubducens ab affirmativâ, habeo pro Areâ, id eſt, pro Logarithmo Hyperbolico Binarii, numerum 6931,4718,0559,9453.

Pro

Pro constructione Tabularum quarumvis numericarum percommoda est Series quæ sequitur. Designent &c. e, d, c, b, a, α, β, γ, δ, ε, &c. terminos alternos in Serie utrinque serpente in infinitum; Pone $A = \alpha + a$, $B = \beta + b$, $C = \gamma + c$, $D = \delta + d$, $E = \varepsilon + e$, &c. Et terminus inter α & a erit

$$\frac{A}{2} +$$

$$\frac{1}{1} \times \frac{A-B}{2^4} +$$

$$\frac{1.3}{1.2} \times \frac{2A-3B+C}{2^7} +$$

$$\frac{1.3.5}{1.2.3} \times \frac{5A-9B+5C-D}{2^{10}} +$$

$$\frac{1.3.5.7}{1.2.3.4} \times \frac{14A-28B+20C-7D+E}{2^{13}} +$$

$$\frac{1.3.5.7.9}{1.2.3.4.5} \times \frac{42A-90B+75C-35D+9E-F}{2^{16}} +$$

$$\frac{1.3.5.7.9.11}{1.2.3.4.5.6} \times \frac{132A-297B+275C-154D+54E-11F+G}{2^{19}} +$$

&c.

Hæc Series sequitur ex casu tertio Propositionis, ponendo $z = 0$. Coefficientes numerales literarum sic producuntur; exempli gratiâ, in quarto termino coefficiens literæ penultimæ C est 5; pone $5 + 1 = n$, & numeri qui proveniunt ex multiplicatione terminorum $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$ &c. erunt 1, 6, 15, 20, &c. Horum differentiæ 5, 9, 5, sunt numeri quæsiti. Atque adeo Series ad libitum produci potest.

Datis Logarithmis numerorum 46, 48, 50, 52, 54, 56, 58 & 60; invenire Logarithmum numeri 53, qui consistit in medio omnium. Pone $l, 52 + l, 54 = A = 3,4483,9710,34$, $l, 50 + l, 56 = B = 3,4471,5803,13$.

$l, 48 + l, 58 = C = 3,4446.6923,08$, $l, 46 + l, 60 = D = 3,4409,0908,19$. Hisce valoribus in Serie ſcriptis, primi quatuor termini dabunt 1,7242,2586,96 pro Logarithmo numeri 53. Et eadem ratione invenire licet quemvis alium intermedium.

In Conſtructione ergo Tabularum ſufficit primo quærere aliquos terminos in debitis diſtantiis nam reliqui poſſunt hoc modo interſeri. Etenim continuo ſunt intercalandi termini primo inventi, uſque dum perventum fuerit ad ultimos qui deſiderantur. Hoc modo habebitur tota Tabula ex datis paucis terminis ſub initio pro fundamento operationis. Sed non convenit ut termini quos primo quærimus, ſint omnes per totam Tabulam æquidiſtantes; nam ſi omittimus alternos ubi eorum differentia eſt maxima, poſſumus alibi per ſaltum omittere duos, tres, viginti aut forte plures terminos. Numerus autem terminorum inter duos datos conſiſtentium, qui omittuntur, debet ſemper eſſe aliquis ſequentium 1, 3, 7, 15, 31, 63, &c. dummodo volumus inſerere eos per hanc Seriem; hoc vero neutiquam incommodabit opus.

Poſſunt autem pro Praxi termini in unam ſummam colligi, ut factum vides in hac Tabella. Prima expreſſio eſt primus terminus; ſecunda eſt ſumma primi & ſecundi; tertia eſt ſumma primi, ſecundi & tertii: & ſic porro.

2	$\frac{A}{2}$
4	$\frac{9A - B}{16}$
6	$\frac{150A - 25B + 3C}{256}$
8	$\frac{1225A - 245B + 49C - 5D}{2048}$
10	$\frac{39690A - 8820B + 2268C - 305D + 35E}{65536}$

Sic

Sic datis aliquibus terminis alternis, intermedii confestim dabuntur per hasce expressiones, nullâ ratione habitâ naturæ Tabulæ particularis. Nam hæ regulæ sunt eædem in omnibus. Areæ curvarum sunt proxime æquales areis Parabolicæ figuræ quæ transit per extrema Ordinatarum suarum. Sed quoniam laboriosum nimis esset semper recurrere ad Parabolam, computavi Tabulam sequentem, quâ Areæ directe exhibentur ex datis Ordinatis.

1	$\frac{A}{1}R$
3	$\frac{A+4B}{6}R$
5	$\frac{7A+32B+12C}{90}R$
7	$\frac{41A+216B+27C+272D}{840}R$
9	$\frac{989A+5888B-928C+10496D-4540E}{28350}R$
11	$\frac{16067A+106300B-48525C+272400D-260550E+427368F}{598752}R.$

Hic numerus Ordinatarum est impar, A est summa primæ & ultimæ, B secundæ & penultimæ, C tertiæ & antepenultimæ; & sic porro, usque dum deventum sit ad eam in medio omnium, quæ per ultimam literam in quâque expressione repræsentatur. R est basis seu pars Abscissæ inter primam & ultimam Ordinatam interceptæ. Expressiones sunt Areæ contentæ inter Curvam, basin & Ordinatas hinc inde extremas. Tabulam pro pare numero Ordinatarum non apposui, quoniam Area cæteris paribus ex impare earum numero accuratius definitur.

Quæratur area quæ generatur ab Ordinatâ $\overline{1+zz}|^{-1}$ & jacet supra Abscissam z quando ea evadit unitas. In

$\overline{1+zz}|^{-1}$, pro z ſcribe $\frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, \frac{10}{10}$; & prodibunt undecim Ordinatæ $1, \frac{100}{101}, \frac{25}{26}, \frac{100}{109}, \frac{25}{29}, \frac{4}{5}, \frac{25}{34}, \frac{100}{149}, \frac{25}{41}, \frac{100}{181}, \frac{1}{2}$. Hinc eſt $A = 1 + \frac{1}{2} = \frac{3}{2}$, $B = \frac{100}{101} + \frac{100}{181} = \frac{28200}{18281}$, $C = \frac{25}{26} + \frac{25}{41} = \frac{1675}{1066}$, $D = \frac{100}{109} + \frac{100}{149} = \frac{25800}{16241}$, $E = \frac{25}{29} + \frac{25}{34} = \frac{1575}{986}$, $F = \frac{4}{5}$. Hiſce valoribus ſubſtitutis in ultimâ expreſſione, & unitate pro R, invenies aream eſſe 785398187. Juſtus eſt hic numerus in ſeptimâ figurâ, in octavâ verum ſuperans Binario.

Si undecim Ordinatæ non dent aream ſatis exactam, erige plures; & concipe aream diviſam eſſe in plures partes, quarum quamque ſeorſum quærens habebis pro lubitu juſtam.

Valor ipſius $\overline{1+Q}|^n$ exprimi poteſt per quamcunque trium ſerierum ſequentium.

$$\overline{1+Q}|^n = 1 +$$
$$Q \times \frac{n}{1} +$$
$$Q^2 \times \frac{n}{1} \times \frac{n-1}{2} +$$
$$Q^3 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} +$$
$$Q^4 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$$
$$Q^5 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} + \&c.$$

$$\text{Vel } \overline{1+Q}|^n = 1 +$$
$$R \times \frac{n}{1} +$$
$$R^2 \times \frac{n}{1} \times \frac{n+1}{2} +$$
$$R^3 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} +$$

$R^4 \times$

$$R^4 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} +$$

$$R^5 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} + \&c.$$

posito scilicet $R = \frac{1+Q}{Q}$. Vel

$$\overline{1+Q}|^n = 1 +$$

$$\frac{2+\overline{n+1}\times Q}{\overline{1+Q}|^1} \times Q \times \frac{n}{1.2} +$$

$$\frac{4+\overline{n+2}\times Q}{\overline{1+Q}|^2} \times Q^3 \times \frac{n}{1.2} \times \frac{nn-1}{3.4} +$$

$$\frac{6+\overline{n+3}\times Q}{\overline{1+Q}|^3} \times Q^5 \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{5.6} +$$

$$\frac{8+\overline{n+4}\times Q}{\overline{1+Q}|^4} \times Q^7 \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{5.6} \times \frac{nn-9}{7.8} +$$

$$\frac{10+\overline{n+5}\times Q}{\overline{1+Q}|^5} \times Q^9 \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{5.6} \times \frac{nn-9}{7.8} \times \frac{nn-16}{9.10}$$

$$+ \&c.$$

Primæ duæ Series demonstrantur per Casum primum Propositionis Nam si $\overline{1+Q}|^0$, $\overline{1+Q}|^1$, $\overline{1+Q}|^2$, $\overline{1+Q}|^3$, $\overline{1+Q}|^4$, *&c.* designent Ordinatas totidem æquidistantes in Parabolicâ figurâ. erit $\overline{1+Q}|^n$ ejusdem Ordinata, cujus distantia à $\overline{1+Q}|^0$ est n. Et sic prodit Series prima. At si in alia Parabola $\overline{1+Q}|^0$, $\overline{1+Q}|^{-1}$, $\overline{1+Q}|^{-2}$, $\overline{1+Q}|^{-3}$, $\overline{1+Q}|^{-4}$, *&c.* sint æquidistantes Ordinatæ, erit $\overline{1+Q}|^n$ Ordinata in eadem, cujus distantia à $\overline{1+Q}|^0$ est $-n$; sic proveniet Series secunda Sit jam in tertia Parabola &c. $\overline{1+Q}|^{-4}$, $\overline{1+Q}|^{-3}$, $\overline{1+Q}|^{-2}$, $\overline{1+Q}|^{-1}$, $\overline{1+Q}|^0$, $\overline{1+Q}|^1$, $\overline{1+Q}|^2$, $\overline{1+Q}|^3$, $\overline{1+Q}|^4$, *&c.* Series Ordinatarum

 æqui-

æquidiſtantium hinc inde progrediens in infinitum, eritque in eadem $\overline{1+Q}|^n$ Ordinata, diſtantiâ n à termino medio $\overline{1+Q}|^0$ remota. Et ſic provenit Series tertia per Caſum Secundum Propoſitionis. Prima abrumpit quando eſt n integer & affirmativus, ſecunda quando eſt n integer & negativus, & tertia in caſu utroque abrumpit. Per harum quamque radices numerales commode evolvuntur in Series. Tertia reliquis multo citius convergit: ejus terminus ſecundus adhiberi poteſt pro correctione, ubi fit extractio per repetitionem calculi.

Halleius in ſua methodo conſtruendi Logarithmos, ex prima harum ſerierum demonſtrat Seriem *Mercatoris* pro Quadratura Hyperbolæ. Sit ejus Ordinata $\overline{1+z}|^{-1}$, vel $\overline{1+z}|^{n-1}$, exiſtente n numero infinite parvo; unde per methodos Quadrandi, area quæ jacet ſupra Abſciſſam z, id eſt, Logarithmus numeri $1+z$, erit $\frac{\overline{1+z}|^n-1}{n}$. Eſt vero per primam Seriem $\overline{1+z}|^n=1+\frac{n}{1}z+\frac{n}{1}\times\frac{n-1}{2}z^2+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}z^3+$ &c. adeoque in caſu præſente, ubi eſt n infinite parvus, eſt $\overline{1+z}|^n=1+\frac{n}{1}z-\frac{n}{2}z^2+\frac{n}{3}z^3-\frac{n}{4}z^4$ &c. quo ſubſtituto in valore areæ, ea prodit $z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4+\frac{1}{5}z^5-$ &c. quæ eſt Series *Mercatoris.*

Similiter per Seriem ſecundam prodit hæc regula; Sit datus numerus $1+z$, pone $R=\frac{z}{1+z}$, eritque ejus Logarithmus $R+\frac{1}{2}R^2+\frac{1}{3}R^3+\frac{1}{4}R^4+\frac{1}{5}R^5+$ &c.

Per Seriem tertiam provenit ſequens regula. Sit quilibet numerus R, pone $z=\frac{\overline{R-1}|^2}{2R}$, eritque ejus Logarithmus

garithmus $\frac{RR-1}{2R} - \frac{1}{3}Az - \frac{2}{5}Bz - \frac{3}{7}Cz - \frac{4}{9}Dz - \frac{5}{11}Ez$ — &c. Ubi $A, B, C, D, E,$ &c. more *Newtoniano* designant terminos Seriei ſicut ab initio. Hæc Series, ut ea ex qua deducitur, reliquis duabus multis vicibus celerius approximat: eſtque eadem generalius expreſſa quam, ex fundamento haud abſimili, pro inventione Logarithmi Binarii prius dedimus.

Methodus inveniendi valores Serierum Arithmeticarum utcunque tarde convergentium.

In aliquibus Seriebus ſumma terminorum haberi nequit niſi ad pauciſſima figurarum loca, dummodo præter ſimplicem eorum additionem aliæ artes non adhibeantur. Proponatur jam Series quælibet cujus termini omnes iiſdem ſignis afficiuntur, & quorum proximi continue tendunt eſſe inter ſe æquales; quales ſunt ſequentes $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} +$ &c. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} +$ &c. Collige ſummam aliquot terminorum ſub initio, ii proxime addendi ſint $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta,$ &c. In numeris proximis ſit $r = \frac{\alpha\gamma - \beta\beta}{\alpha\beta - 2\alpha\gamma + \beta\gamma}$, & quantitatum $\alpha \times \frac{\alpha + r\beta}{\alpha - \beta}$, $\alpha + \beta \times \frac{\beta + r\gamma}{\beta - \gamma}$, $\alpha + \beta + \gamma \times \frac{\gamma + r\delta}{\gamma - \delta}$, $\alpha + \beta + \gamma + \delta \times \frac{\delta + r\varepsilon}{\delta - \varepsilon}$, $\alpha + \beta + \gamma + \delta + \varepsilon \times \frac{\varepsilon + r\zeta}{\varepsilon - \zeta}$, &c. differentiæ ſint $a, b, c, d, e,$ &c. Deinde in numeris proximis ſit $s = \frac{ac - bb}{ab - 2ac + bc}$, & ipſorum $a \times \frac{a + sb}{a - b}$, $a + b \times \frac{b + sc}{b - c}$, $a + b + c \times \frac{c + sd}{c - d}$, $a + b + c + d \times \frac{d + se}{d - e}$, &c. differentiæ ſint $A, B, C, D,$ &c. & ſit $t = \frac{AC - BB}{AB - 2AC + BC}$: atque ſic procede

cede quoad libuerit. Tum erit $\alpha + \beta + \gamma + \delta + \epsilon + \&c. = \alpha \times \frac{\alpha + r\beta}{\alpha - \beta} + a \times \frac{a + sb}{a - b} + A \times \frac{A + tB}{A - B} + \&c.$ atque ultra duos primos terminos hujus novæ Seriei raro opus erit progredi.

Ut si desideretur valor Seriei $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \&c.$ collige primos 21 terminos, quorum summam reperio fore 6813,8410,1885. Termini proxime addendi sunt $\alpha =$,0005,2854,1226, $\beta =$,0004,8309,1787, $\gamma =$,0004,4326,2411, $\delta =$,0004,0816,3265, &c. Hinc fit $r = 1$ proxime, & $\alpha \times \frac{\alpha + r\beta}{\alpha - \beta}$ =,0117,6449,6282, $a =$ —,0000,0017,5096, $b =$ —,0000,0014,7410, $c =$ —,0000,0012,4986, &c. Unde $s = \frac{1}{5}$ prope, & $a \times \frac{a + sb}{a - b} =$ —,0000,0141,8111, quem propter signum negativum subduco ab $\alpha \times \frac{\alpha + r\beta}{\alpha - \beta}$, & remanet, 0117,6307, 8171: hic additus summæ primo inventæ 6813,8410, 1885, dat pro summa totius Seriei numerum 6931, 4718,0056, qui justus est in nonâ decimali; at ante duas hasce correctiones summa erat justa in primâ figura solâ. Si animus sit propius scopum attingere, pergendum erit ad approximationes sequentes. Si termini Seriei diversa habeant signa, conjungendi sunt, ut omnes eadem tandem habeant, ut in Serie $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$ conjunctis terminis ea evadit $\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15} + \&c.$ Sed hic notandum est quod differentiæ $a, b, c, d, e,$ &c. ut & $A, B, C, D,$ &c. colligi debent subducendo quantitates antecedentes de subsequentibus. Et in omnibus hujusmodi Seriebus si $p, q, r,$ repræsentent tres terminos ordine sequentes, p primum,

mum, q ſecundum, r tertium, & rectangulum $\frac{p+r}{2} \times q$ non ſit majus pr, valor Seriei erit infinite magnus: at magnitudinis ſemper finitæ ubi accidit contrarium. Poteſt hæc regula nonunquam fallere, ubi termini p, q, r parum diſtant ab initio Seriei, at ſi conſiſtant inter eos ab initio aliquantum remotos, evadet regula certiſſima.

Ad alia Serierum genera debent aliæ regulæ adhiberi. Sit Series regularium Polygonorum Circulo Incriptorum, exiſtente Radio unitate.

$H =$	2,0000,0000,0000,000	4
$G =$	2,8284,2712,4746,190	8
$F =$	3,0614,6745,8920,718	16
$E =$	3,1214,4515,2258,051	32
$D =$	3,1365,4849,0545,938	64
$C =$	3,1403,3115,6954,752	128
$B =$	3,1412,7725,0932,772	256
$A =$	3,1415,1380,1144,299	512

Dicatur jam ultimum Polygonum A, penultimum B, antepenultimum C, & reliqua in ſuo ordine retrorſum D, E, F, &c. atque area Circuli quæſita erit $A + \frac{A-B}{3} + \frac{4A-5B+C}{3 \cdot 15} + \frac{64A-84B+21C-D}{3 \cdot 15 \cdot 63} + \frac{4096A-5440B+1428C-85D+E}{3 \cdot 15 \cdot 63 \cdot 255} +$ &c. Ubi ſi pro A, B, C, D, E, &c. ſcribantur proprii valores, primi quatuor termini dabunt 3,1415,9265,3589,790 pro area Circuli. Hæc autem Series eſt generalis, ex natura Circuli neutiquam dependens: applicabilis eſt quotieſcunque numerorum approximantium differentiæ priores ſunt poſteriorum quaſi quadruplæ. Factores in Denominatoribus ſunt dignitates integræ numeri 4 unitatibus minutæ:

nuræ: quibus datis, coefficientes literarum in diversis terminis formantur ex multiplicatione continua numerorum $1 . \frac{n}{3}, \frac{n-3}{15}, \frac{n-15}{63}, \frac{n-63}{255}$, &c. Ubi pro n substituendus est ultimus Factorum in Denominatore.

Ultima quantitatum $x-1$, $2\sqrt[2]{x}-2$, $4\sqrt[4]{x}-4$, $8\sqrt[8]{x}-8$, $16\sqrt[16]{x}-16$, &c. æqualis est Logarithmo numeri x. Pro x scribe 2, & per repetitam extractionem radicis quadratæ exibunt numeri

$$
\begin{aligned}
M &= 1{,}0000{,}0000{,}0000{,}0000. \\
L &= 8284{,}2712{,}4746{,}1901. \\
I &= 7568{,}2864{,}0010{,}8843. \\
H &= 7240{,}6186{,}1322{,}0613. \\
G &= 7083{,}8051{,}8838{,}6214. \\
F &= 7007{,}0875{,}6931{,}7337. \\
E &= 6969{,}1430{,}7308{,}8294. \\
D &= 6950{,}2734{,}2438{,}7611. \\
C &= 6940{,}8641{,}2851{,}8363. \\
B &= 6936{,}1658{,}4759{,}4014. \\
A &= 6933{,}8182{,}9699{,}9493.
\end{aligned}
$$

Dicatur ultimus numerorum A, penultimus B, & sic retro, atque Logarithmus quæsitus erit $A + \frac{A-B}{1} + \frac{2A-3B+C}{1 . 3} + \frac{8A-14B+7C-D}{1 . 3 . 7} + \frac{64A-120B+70C-15D+E}{1 . 3 . 7 . 15}$ + &c. Primi quinque termini dant 6931,4718,0559, 9457 pro Logarithmo Hyperbolico Binarii. Et quomodo hæc Series procedit in infinitum facile colligitur ex eo quod de priore diximus: estque etiam universalis, proprietates Hyperbolæ minime respiciens.

Extenditur quoque Methodus hæcce Differentialis ad Resolutionem Æquationum & alia quamplurima quorum hic non fit mentio. Continetque fundamenta Serierum generalissima; ut in Reductione Æquationum Irrationalium & Fluxionalium brevi forsan monstrabo.

IV. *An*